AN EXISTENCE AND NON-UNIQUENESS THEOREM OF VORTEX WAVES OF THE PERIODIC TYPE

(TEOREMA SUSHCHESTVOVANIIA I NEEDINSVENNOSTI VIKHREVYKH VOLN PERIODICHESKOGO TIPA)

PMM Vol. 24, No. 4, 1960, pp. 711-714

N. N. MOISEEV
(Moscow)

(Received 13 April 1960)

1. The determination of the standing gravitational waves on the surface of a rotational fluid reduces to the following boundary value problem:

$$\Delta\psi = F(\psi) \text{ in the region } T$$
$$\psi(x, 0) = 0, \qquad \psi(x, f) = 1$$
$$\psi_x{}^2 + \psi_y{}^2 + 2\nu f = 0 \quad \text{for } y = f \qquad \left(\nu = \frac{gh^3}{Q^2},\ h = \frac{Q}{c}\right) \tag{1}$$

where $y = f(x)$ is the equation of the unknown free surface (figure), ψ is the stream function, h is the "depth" of the fluid, Q is the flow rate and c is the velocity.

In the relationships (1) all the unknowns are assumed to be dimensionless, Q and h are taken to be the characteristic dimensions.

The formulation and the first results of the problem (1) are due to Dubreil-Jacotoin [1]. The most general results published up to the present time are due to Gouyon [2, 3]. His basic assumptions are the following: $F(\psi)$ is a continuous function and the flow is near the uniform flow $\psi = y$.

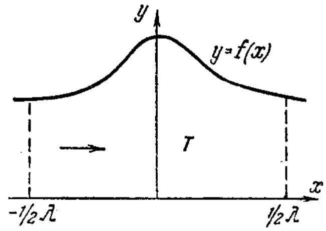
2. If we select x and ψ as independent variables, while the dependent variables are $u = \psi_y$ and $v = -\psi_x$, then the problem reduces to the following [2]:

$$uv_\psi - vu_\psi + u_x = 0, \qquad uu_\psi + vv_\psi - v_x = F(\psi)$$
$$v = 0 \quad \text{for } \psi = 0,$$
$$uu_x + vv_x + \nu v/u = 0 \quad \text{for } \psi = 1 \tag{2}$$

Problem (2) always admits the solution

$$v = 0, \qquad u \equiv z(\psi) = \left(2\int_0^\psi F(\xi)\,d\xi + 1\right)^{1/2}$$

Let us assume that the function $F(\psi)$ is such that $z(\psi) \geqslant d > 0$, where d is a positive constant.



We shall introduce new variables

$$u = ze^{\tau}\cos\theta, \qquad v = ze^{\tau}\sin\theta \qquad \left(\tau = \frac{1}{2}\ln\frac{u^2+v^2}{z^2}\right)$$

where θ is the angle of inclination of the velocity vector. In terms of these variables, problem (2) is equivalent to the following problem:

$$z\theta_\psi + \tau_x = \Phi_1(\theta, \tau), \qquad z\tau_\psi - \theta_x = \Phi_2(\theta, \tau) \tag{3}$$

$$\theta = 0 \qquad \text{for } \psi = 0 \tag{4}$$

$$\tau_x{}^* = -\nu e^{-2\tau^*}\frac{\tan\theta^*}{z^2(1)} \tag{5}$$

where the star denotes that the values of τ and θ are finite. Φ_1 and Φ_2 are the nonlinear operators

$$\Phi_1(\theta, \tau) = -z\theta_\psi(e^\tau - 1) - \tau_x(\cos\theta - 1) + \theta_x\sin\theta$$

$$\Phi_2(\theta, \tau) = \frac{F(\psi)}{z(\psi)}(e^{-\tau} - e^\tau) - z\tau_\psi(e^\tau - 1) + \theta_x(\cos\theta - 1) + \tau_x\sin\theta$$

The problem (3)-(5) has a trivial solution $\tau \equiv \theta \equiv 0$. We shall state the problem of finding the periodic solutions in x of the period λ (dimensional wavelength) of problem (3)-(5) adding to the enumerated conditions the condition of symmetry

$$\theta(-{}^1/_2\lambda) = \theta({}^1/_2\lambda) = 0 \tag{6}$$

3. We shall investigate an auxiliary problem, in which we shall replace condition (5) by the following condition:

$$\tau(x, 1) \equiv \tau^*(x) = f_1(x) \tag{7}$$

Assume that $\theta = \theta_1 + \theta_2$, $\tau = \tau_1 + \tau_2$, where τ_1 and θ_1 is the solution of the boundary-value problem (4), (6) and (7) for the system (Problem A)

$$z\theta_\psi + \tau_\psi = 0, \qquad z\tau_\psi - \theta_x = 0 \tag{8}$$

where θ_2 and τ_2 solve a homogeneous boundary problem for system (3), in which the exprėssions of the right-hand terms are replaced by the following expressions (problem B):

$$\Phi_i\,(\theta_1 + \theta_2,\ \tau_1 + \tau_2)$$

Lemma 1. The solution of problem A has the form

$$\theta_1 = A_1\,(\xi), \qquad \tau_1 = B_1\,(\xi) + \tau_0 \qquad \left(\xi = \frac{d\tau^*}{dx}\right) \tag{9}$$

where the operators A_1 and B_1 are linear integral operators with weak singularity and $\tau_0 = \tau^*(0)$.

Lemma 2. Problem B is equivalent to the following system of equations:

$$\theta_2\,(x, \psi) = -\int_0^{1/2\lambda}\int_0^1 K_{11}\,(x, \psi;\ x,'\,\psi')\,\Phi_1\,(\theta_1 + \theta_2,\ \tau_1 + \tau_2)\,dx'd\psi' -$$
$$-\int_0^{1/2\lambda}\int_0^1 K_{12}\Phi_2\,\{\theta_1\,(x,'\,\psi') + \theta_2\,(x',\psi');\ \tau_1 + \tau_2\}\,dx'd\psi' \tag{10}$$
$$\tau_2\,(x, \psi) = -\int_0^{1/2\lambda}\int_0^1 K_{21}\Phi_1\,dx'\,d\psi' + \int_0^{1/2\lambda}\int_0^1 K_{22}\Phi_2\,dx'\,d\psi'$$

where

$$K_{11} = \sum_{n,\,m} \frac{\mu_m}{\mu_m{}^2 + (\alpha n)^2}\,\frac{2\alpha \sin n\alpha x \sin n\alpha x'}{\pi}\,\frac{\chi_m\,(\psi)\,\chi_m\,(\psi')}{z\,(\psi')}$$
$$K_{12} = \sum_{n,\,m} \frac{\alpha n}{\mu_m{}^2 + (\alpha n)^2}\,\frac{2\alpha \sin n\alpha x \cos n\alpha x'}{\pi}\,\frac{\chi_m\,(\psi)\,\chi_m\,(\psi')}{z\,(\psi')}$$
$$K_{21} = \sum_{n,\,m} \frac{\alpha n}{\mu_m{}^2 + (\alpha n)^2}\,\frac{2\alpha \cos n\alpha x \sin n\alpha x'}{\pi}\,\frac{g_m\,(\psi)\,g_m\,(\psi')}{z\,(\psi')}$$
$$K_{22} = \sum_{n,\,m} \frac{\mu_m}{\mu_m{}^2 + (\alpha n)^2}\,\frac{2\alpha \cos n\alpha x \cos n\alpha x'}{\pi}\,\frac{g_m\,(\psi)\,\chi_m\,(\psi')}{z\,(\psi')}$$

where $a = 2\pi/\lambda$, g_n and μ_n are the eigenfunctions and eigenvalues of the operator

$$\frac{d}{d\psi}\left(z\,\frac{dg_n}{d\psi}\right) = -\,\mu_n{}^2\,\frac{g_n}{z}\,, \qquad g\,(1) = \left(\frac{dg}{d\psi}\right)_{\psi=0} = 0, \qquad \chi_n = \frac{z}{\mu_n}\,\frac{dq_n}{d\psi}$$

The functions g_n and x_n are normalized with the weighting function $1/z$.

Lemma 3. For any differentiable τ_1 and θ_1, which satisfy the conditions $|\tau_1| < \epsilon$, $|\tau_{1x}| < \epsilon_1$ ..., $|\theta_{1\psi}| < \epsilon$ where ϵ is some sufficiently small positive number, system (10) has the unique solution

$$\theta_2 = A_2(\theta_1, \tau_1), \qquad \tau_2 = B_2(\theta_1, \tau_1) \tag{11}$$

where A_2 and B_2 are integer-power series of their variables; they converge uniformly in the rectangle T

$$-\tfrac{1}{2}\lambda \leqslant x \leqslant \tfrac{1}{2}\lambda, \qquad 0 \leqslant \psi \leqslant 1$$

The validity of this lemma follows from the theory of integral-differential equations developed in (5), a particular case of which is the system (10). In addition, the proof is based on the fact that the homogeneous boundary-value problem (4), (6) and (7) for the system (3) has only a trivial solution.

We shall denote by A_i^* the operators A_i under the condition that $\psi = 1$. Then

$$\theta_2^* = A_2^*(A, \xi) = D\xi \tag{12}$$

On the basis of the above statements $D\xi$ is an integral-power series, which converges uniformly on $-1/2\,\lambda$, $1/2\,\lambda$, if the magnitude of ξ is sufficiently small. Substituting the expressions obtained in Equation (5), we obtain

$$\xi = -\frac{\nu}{z^2}\exp(-2\tau_0)\exp\left(-2\int_0^x \xi\,dx'\right) \tag{13}$$

The function $\exp(-2\tau_0)$ is analytic. We shall denote

$$k = -\nu\exp(-2\tau_0)$$

Then the problem will reduce to an equation of the form

$$\xi = kR\xi \tag{14}$$

In accordance with the foregoing, the operator R is the Liapunov operator, and, consequently, the general theory shows that Equation (14) has nontrivial solutions with a small norm in the region of single-valued eigenvalues of the corresponding linear problem. Hence, taking into consideration the structure of the function k, we arrive at the following basic theorem:

Theorem. If the function $F(\psi)$ is such that the integral

$$\int_0^1 F(\psi)\,d\psi \qquad (z \geqslant d > 0)$$

exists, then for sufficiently small $\epsilon > 0$ the problem (3)-(5) for a fixed value of the period λ has a one-parametrical family of solutions, if only

$$\nu_n - \nu < \varepsilon$$

where ν_n are the eigen numbers of the linearized problem.

Notes. 1. To compute the wave parameters it is not necessary actually to construct the operator R. It may be shown that the solution of the problem (3)-(5) is an analytic function of the parameter $\sqrt{\nu_n - \nu}$. Therefore, it is simpler to look directly for the solution of the resulting boundary-value problem in the form of a power series in this parameter.

2. The obtained solution will approximate not the uniform flow, as was the case in the prvious investigations, but some rotational flow. For $F \equiv 0$ the obtained results lead to the classical results of Nekresov-Levi-Civita.

3. For the determination of the function τ_0 the following equation may be used:

$$1 = \int_0^1 (u)_{x=0}\,d\psi = \int_0^1 z(\psi)\exp\tau(0,\psi)\,d\psi$$

where

$$\tau = \tau_0 + B_1(\xi) + B_2(A_1(\xi), \qquad \tau_0 + B_1(\xi))$$

4. The operators A_1 and B_1 have the following form:

$$\theta_1 = A_1(\xi) = \int_0^{1/2\pi} \xi(x') \sum \frac{2\alpha \sin n\alpha x' \sin n\alpha x}{\pi n\alpha}\,\frac{z(1)\,g_n^{*\prime}(1)}{n\alpha}\,dx'$$

$$\tau_1' = \tau_0 + B_1(\xi) = \tau_0 + \int_0^{1/2\pi} \xi(x') \sum \frac{2\alpha \cos n\alpha x \sin n\alpha x'}{\pi n\alpha}\,g_n^*(1)\,dx'$$

where g_n are functions which satisfy the equation

$$z\frac{d}{d\psi}(zg_n^*) - n^2\alpha^2 g_n^* = 0$$

and the condition $g_n'(0) = 0$. The functions g_n^* have the form

$$g_n^*(\psi) = \cosh\left(n\alpha\int_0^{\psi}\frac{d\psi}{z(\psi)}\right)$$

5. The functions g_n introduced in Lemma 2 have the form

$$g_n(\psi) = \delta \cos \mu_n \int_0^1 \frac{d\psi}{z(\psi)} \left(\mu_n = n\pi \left[\int_0^z \frac{d\psi}{z(\psi)}\right]^{-1}\right)$$

where δ is a normalizing multiplier.

Consequently, all computations may actually be carried out for any given function $F(\psi)$.

In conclusion I consider it to be my pleasant duty to express my gratitude to Iu.A. Kravchenko (France, Grenoble), who directed my attention to these problems and whose discussions motivated this investigation.

BIBLIOGRAPHY

1. Dubreil-Jacotoin, M.L., Sur la determination rigoreuse des ondes permanents periodiques d'ampleur finie. *J. Math. Pures et Appl.* Vol. 13, 1934.

2. Gouyon, R., Sur les houles planes en profondeur infinie. *C.R. Acad. Sci., Paris* Vol. 247, pp. 33-35, 1958.

3. Gouyon, R., Sur les houles planes en profondeur finie. *C.R. Acad. Sci., Paris* Vol. 247, pp. 180-182, 1958.

4. Moiseev, N.N., O techenii tiazheloi zhidkosti nad velnistym dnom (On the flow of heavy fluid above a wavy bottom). *PMM* Vol. 21, No. 1, 1957.

5. Lichtenstein, *Vorlesungen ueber nichtlinearen Integral und Integro-differenzial Gleichungen.* Berlin, 1931.

Translated by J.R.W.